

Reformulations of the Yang-Mills theory toward quark confinement and mass gap

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Abstract. We propose the reformulations of the $SU(N)$ Yang-Mills theory toward quark confinement and mass gap. In fact, we have given a new framework for reformulating the $SU(N)$ Yang-Mills theory using new field variables. This includes the preceding works given by Cho, Faddeev and Niemi, as a special case called the maximal option in our reformulations. The advantage of our reformulations is that the original non-Abelian gauge field variables can be changed into the new field variables such that one of them called the restricted field gives the dominant contribution to quark confinement in the gauge-independent way. Our reformulations can be combined with the $SU(N)$ extension of the Diakonov-Petrov version of the non-Abelian Stokes theorem for the Wilson loop operator to give a gauge-invariant definition for the magnetic monopole in the $SU(N)$ Yang-Mills theory without the scalar field. In the so-called minimal option, especially, the restricted field is non-Abelian and involves the non-Abelian magnetic monopole with the stability group $U(N-1)$. This suggests the non-Abelian dual superconductivity picture for quark confinement. This should be compared with the maximal option: the restricted field is Abelian and involves only the Abelian magnetic monopoles with the stability group $U(1)^{N-1}$, just like the Abelian projection. We give some applications of this reformulation, e.g., the stability for the homogeneous chromomagnetic condensation of the Savvidy type, the large N treatment for deriving the dimensional transmutation and understanding the mass gap, and also the numerical simulations on a lattice which are given by Dr. Shibata in a subsequent talk.

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INTRODUCTION

We follow the Wilson criterion for quark confinement, i.e., area law of the Wilson loop [Wilson (1974)][1]. Therefore, we start from the Wilson loop operator. For a given closed path C , we define the **Wilson loop operator** $W_C[\mathcal{A}]$ for the **non-Abelian Yang-Mills field** $\mathcal{A}_\mu(x)$ by [Yang & Mills (1954)][2]

$$W_C[\mathcal{A}] := \text{tr} \left[\mathcal{P} \exp \left\{ i g_{\text{YM}} \oint_C dx^\mu \mathcal{A}_\mu(x) \right\} \right] / \text{tr}(\mathbf{1}), \quad \mathcal{A}_\mu(x) = \mathcal{A}_\mu^A(x) T_A, \quad (1)$$

where \mathcal{P} denotes the **path-ordering** prescription. In the Yang-Mills theory, we consider the **Wilson loop average** $W(C)$, i.e., a vacuum expectation value of the Wilson loop operator $W_C[\mathcal{A}]$ for a closed loop C :

$$W(C) = \langle W_C[\mathcal{A}] \rangle_{\text{YM}}. \quad (2)$$

For a rectangular loop C of side lengths T and r , the Wilson loop average $W(C)$ is related to a **static quark-antiquark potential** $V_{q\bar{q}}(r)$ as (See the left panel of Fig. 1)

$$W(C) \sim \exp[-TV_{q\bar{q}}(r)], \quad (T \gg r). \quad (3)$$

$W_C[\mathcal{A}]$ is gauge invariant. Therefore, $V_{q\bar{q}}(r)$ is obtained in the gauge-independent way from

$$V_{q\bar{q}}(r) = \lim_{T \rightarrow \infty} \frac{-1}{T} \ln W(C). \quad (4)$$

The numerical simulations exhibit that the static quark-antiquark potential $V_{q\bar{q}}(r)$ is well fitted by the form of the Cornell type: Coulomb+Linear (See the right panel of Fig. 1)

$$V_{q\bar{q}}(r) = -\frac{\alpha}{r} + \sigma r + c,$$

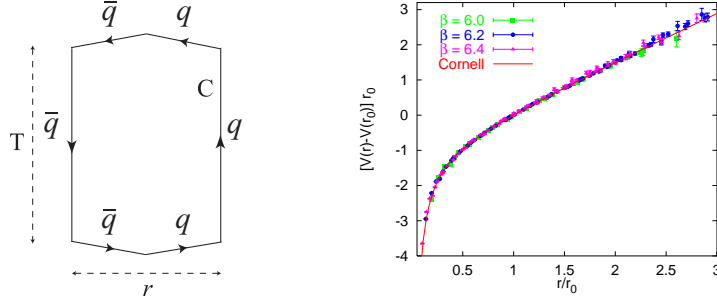


FIGURE 1. (Left) The Wilson loop for a rectangular loop C . (Right) The static quark-antiquark potential $V(r)$ as a function of the distance r in $SU(3)$ Yang-Mills theory, which is obtained by numerical simulations in the framework of lattice gauge theory. Note that the potential is normalized so that $V(r_0) = 0$ and $\beta = 2N/g_{\text{YM}}^2$ for $SU(N)$. See G.S. Bali, [hep-ph/0001312], Phys.Rept.**343**, 1 (2001).

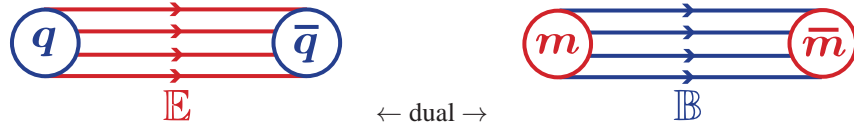


FIGURE 2. The electro-magnetic duality: electric charge is replaced by the magnetic charge, and the electric field is replaced by the magnetic field, and vice versa.

with the three parameters of different dimensions, σ : string tension [mass^2], α : dimensionless [mass^0], and c : [mass^1].

- $\sigma \neq 0$ confinement $V_{q\bar{q}}(r) \rightarrow \infty$ as $r \rightarrow \infty$
- $\sigma = 0$ deconfinement $V_{q\bar{q}}(r) < \infty$ as $r \rightarrow \infty$

The emergence of the dimensionful string tension σ is quite nontrivial, since the Yang-Mills theory includes the dimensionless parameters alone in the classical level.

A promising scenario for understanding quark confinement is called the **dual superconductor hypothesis for quark confinement** based on the electro-magnetic duality (See Fig. 2) proposed by [Nambu (1974), 't Hooft (1975), Mandelstam (1976), and Polyakov (1975,1977) [3]. The key ingredients for the dual superconductivity are as follows. See [7, 6] for reviews.

- **dual Meissner effect**

In the dual superconductor, chromoelectric flux is squeezed into tubes.

[← In the ordinary superconductor, magnetic flux is squeezed into tubes]

- **condensation of chromomagnetic monopoles**

The dual superconductivity is caused by condensation of chromomagnetic monopoles.

[← The ordinary superconductivity is caused by condensation of electric charge into Cooper pairs.]

In order to establish the dual superconductivity, we must answer the following questions:

- * How to introduce magnetic monopoles in the Yang-Mills theory without scalar fields? [This should be compared with the 't Hooft-Polyakov magnetic monopole.]
- * How to define the duality in the non-Abelian gauge theory?

NON-ABELIAN STOKES THEOREM (1)

In order to answer the first question, we consider how the Wilson loop can be related to the magnetic monopole.

First, we consider the Abelian case. The Abelian Wilson loop operator $W_C[A]$ for a loop C is cast into the surface integral over the surface Σ_C bounded by C using the **Stokes theorem**: (See Fig. 3)

$$W_C[A] = \exp \left[ie \oint_C dx^\mu A_\mu \right] \implies W_C[A] = \exp \left[ie \int_{\Sigma_C: \partial \Sigma_C = C} dS^{\mu\nu}(x(\sigma)) F_{\mu\nu}(x(\sigma)) \right]. \quad (5)$$

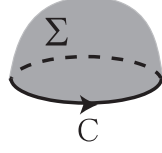


FIGURE 3. A closed loop C for defining the Wilson loop operator and the surface Σ_C whose boundary is given by the loop C .

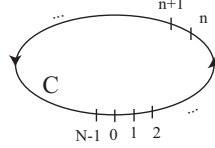


FIGURE 4. A loop C for the Wilson loop is divided into N infinitesimal segments to obtain the path-integral representation.

Introduce the antisymmetric tensor $\Theta_{\Sigma_C}^{\mu\nu}$ called the **vorticity tensor** with the support only on the surface Σ_C bounded by the loop C :

$$\Theta_{\Sigma_C}^{\mu\nu}(x) := \int_{\Sigma_C: \partial \Sigma_C = C} d^2 S^{\mu\nu}(x(\sigma)) \delta^D(x - x(\sigma)). \quad (6)$$

Then the surface integral is rewritten into the spacetime integral over the D -dimensional spacetime:

$$W_C[A] = \exp \{ ie(\Theta_{\Sigma_C}, F) \}, \quad (\Theta_{\Sigma_C}, F) := \int d^D x \Theta_{\Sigma_C}^{\mu\nu}(x) F_{\mu\nu}(x). \quad (7)$$

The Hodge decomposition can be used to define the electric current j and the **magnetic current** k :

$$W_C[A] = \exp \{ ie(N_{\Sigma_C}, j) + ie(\Xi_{\Sigma_C}, k) \}, \quad N_{\Sigma_C} := \delta \Delta^{-1} \Theta_{\Sigma_C}, \quad \Xi_{\Sigma_C} := \delta \Delta^{-1*} \Theta_{\Sigma_C}. \quad (8)$$

The electric current j is non-vanishing: $j := \delta F \neq 0$, while the magnetic current k is vanishing due to the Bianchi identity and there is no magnetic contribution to the Wilson loop:

$$k := \delta^* F = {}^* dF = {}^* ddA = 0 \implies W_C[A] = \exp \{ ie(N_{\Sigma_C}, j) \}, \quad (9)$$

as far as there are no singularities in A .

Next, we consider the non-Abelian case. The non-Abelian Wilson loop operator $W_C[\mathcal{A}]$ (in the representation R) is written using the trace and the path ordering as

$$W_C[\mathcal{A}] := \text{tr}_R \left\{ \mathcal{P} \exp \left[-ig_{\text{YM}} \oint_C \mathcal{A} \right] \right\} / \text{tr}_R(\mathbf{1}). \quad (10)$$

The **path ordering** \mathcal{P} is defined by dividing the path C into N infinitesimal segments (See Fig. 4):

$$W_C[\mathcal{A}] = \lim_{N \rightarrow \infty, \varepsilon \rightarrow 0} \text{tr}_R \left\{ \mathcal{P} \prod_{n=0}^{N-1} \exp \left[-ig_{\text{YM}} \int_{x_n}^{x_{n+1}} \mathcal{A} \right] \right\} / \text{tr}_R(\mathbf{1}). \quad (11)$$

The troublesome path ordering in the non-Abelian Wilson loop operator can be removed as first shown for $G = SU(2)$ by [Diakonov and Petrov (1989)][18], which we call the **non-Abelian Stokes theorem** (NAST). Moreover, the non-Abelian Stokes theorem for the Lie group G can be obtained as the path-integral representation of the Wilson loop operator using the coherent state of the Lie group G in a unified way. [Kondo (1998), Kondo and Taira (2000), Kondo (2008)][19, 20, 21, 22].

In order to derive the non-Abelian Stokes theorem, we follow the standard steps for the path integral:

1. We replace the trace of the operator \mathcal{O} by the integral:

$$\text{tr}_R(\mathcal{O})/\text{tr}_R(\mathbf{1}) = \int d\mu(g(x_0)) \langle g(x_0), \Lambda | \mathcal{O} | g(x_0), \Lambda \rangle, \quad (12)$$

where $d\mu(g)$ is an invariant measure on G and the state is normalized $\langle g(x_n), \Lambda | g(x_n), \Lambda \rangle = 1$.

2. We insert a **complete set** of states at each partition point:

$$\mathbf{1} = \int d\mu(g(x_n)) |g(x_n), \Lambda\rangle \langle g(x_n), \Lambda| \quad (n = 1, \dots, N-1). \quad (13)$$

Here the state $|g, \Lambda\rangle$ is constructed by operating a group element $g \in G$ to a **reference state** $|\Lambda\rangle$ (e.g., the highest-weight state) for a given representation R of the Wilson loop we consider:

$$|g, \Lambda\rangle = g |\Lambda\rangle, \quad g \in G. \quad (14)$$

3. We take the limit $N \rightarrow \infty$ and $\varepsilon \rightarrow 0$ appropriately such that $N\varepsilon$ is fixed:

$$W_C[\mathcal{A}] = \lim_{N \rightarrow \infty, \varepsilon \rightarrow 0} \prod_{n=0}^{N-1} \int d\mu(g(x_n)) \prod_{n=0}^{N-1} \langle g(x_{n+1}), \Lambda | \exp \left[-ig_{\text{YM}} \int_{x_n}^{x_{n+1}} \mathcal{A} \right] | g(x_n), \Lambda \rangle. \quad (15)$$

For taking the limit $\varepsilon \rightarrow 0$ in the final step, it is sufficient to retain the $O(\varepsilon)$ terms:

$$\begin{aligned} & \langle g_{n+1}, \Lambda | \exp \left[-ig_{\text{YM}} \int_{x_n}^{x_{n+1}} \mathcal{A} \right] | g_n, \Lambda \rangle \\ &= \langle \Lambda | g(x_{n+1})^\dagger \exp \left[-ig_{\text{YM}} \int_{x_n}^{x_{n+1}} \mathcal{A} \right] g(x_n) | \Lambda \rangle = \langle \Lambda | \exp \left[-ig_{\text{YM}} \int_{x_n}^{x_{n+1}} \mathcal{A}^g \right] | \Lambda \rangle \\ &= \langle \Lambda | \left[1 - ig_{\text{YM}} \int_{x_n}^{x_{n+1}} d\tau \mathcal{A}^g(\tau) + O(\varepsilon^2) \right] | \Lambda \rangle \\ &= 1 - ig_{\text{YM}} \int_{x_n}^{x_{n+1}} \langle \Lambda | \mathcal{A}^g | \Lambda \rangle + O(\varepsilon^2) \quad (\langle \Lambda | \Lambda \rangle = 1) \\ &= \exp \left[-i\varepsilon g_{\text{YM}} \int_{x_n}^{x_{n+1}} \langle \Lambda | \mathcal{A}^g | \Lambda \rangle \right] + O(\varepsilon^2). \end{aligned} \quad (16)$$

Here $\mathcal{A}^g(x)$ agrees with the gauge transformation of $\mathcal{A}(x)$ by the group element g :

$$\mathcal{A}^g(x) := g(x)^\dagger \mathcal{A}(x) g(x) + ig_{\text{YM}}^{-1} g(x)^\dagger dg(x). \quad (17)$$

Defining the one-form A^g from the Lie algebra valued one-form \mathcal{A}^g by

$$A^g := \langle \Lambda | \mathcal{A}^g | \Lambda \rangle, \quad (18)$$

we arrive at a path-integral representation of the Wilson loop operator (pre-NAST):

$$W_C[\mathcal{A}] = \int [d\mu(g)]_C \exp \left(-ig_{\text{YM}} \oint_C A^g \right), \quad [d\mu(g)]_C := \lim_{N \rightarrow \infty, \varepsilon \rightarrow 0} \prod_{n=0}^{N-1} d\mu(g(x_n)). \quad (19)$$

The path-ordering has disappeared.

Therefore, we can apply the (usual) Stokes theorem to obtain a **non-Abelian Stokes theorem**:

$$W_C[\mathcal{A}] = \int [d\mu(g)]_{\Sigma_C} \exp \left[-ig_{\text{YM}} \int_{\Sigma_C: \partial \Sigma_C = C} F^g \right], \quad F^g = dA^g. \quad (20)$$

Here we have replaced the integration measure on the loop C by the integration measure on the surface Σ_C :

$$[d\mu(g)]_{\Sigma_C} := \prod_{x \in \Sigma_C: \partial \Sigma_C = C} d\mu(g(x)), \quad (21)$$

by inserting additional integration measures, $1 = \int d\mu(g(x))$ for $x \in \Sigma_C - C$. The explicit expression for F^g will be obtained later.

FIELD DECOMPOSITION FOR $SU(2)$ A LA CHO-DUAN-GE-FADDEEV-NIEMI

For the **highest-weight state** $|\Lambda\rangle = (\lambda_a)$ of a representation R of a group G , we define a matrix ρ with the matrix element ρ_{ab} by

$$\rho := |\Lambda\rangle \langle \Lambda|, \quad \rho_{ab} := |\Lambda\rangle_a \langle \Lambda|_b = \lambda_a \lambda_b^*. \quad (22)$$

Since $|\Lambda\rangle$ is normalized: $\langle \Lambda|\Lambda\rangle = \lambda_a \lambda_a^* = 1$, the trace of ρ has a unity:

$$\text{tr}(\rho) = \rho_{aa} = 1. \quad (23)$$

Moreover, the matrix element $\langle \Lambda|\mathcal{O}|\Lambda\rangle$ of an arbitrary matrix \mathcal{O} is written in the trace form:

$$\langle \Lambda|\mathcal{O}|\Lambda\rangle = \text{tr}(\rho \mathcal{O}), \quad (24)$$

since $\langle \Lambda|\mathcal{O}|\Lambda\rangle = \lambda_b^* \mathcal{O}_{ba} \lambda_a = \rho_{ab} \mathcal{O}_{ba} = \text{tr}(\rho \mathcal{O})$.

By using the operator ρ , the ‘‘Abelian’’ field A^g is written in the trace form of a matrix:

$$A^g(x) = \langle \Lambda|\mathcal{A}^g(x)|\Lambda\rangle = \text{tr}\{\rho \mathcal{A}^g(x)\} = \text{tr}\{g(x)\rho g^\dagger(x)\mathcal{A}(x)\} + ig_{\text{YM}}^{-1} \text{tr}\{\rho g^\dagger(x)dg(x)\}. \quad (25)$$

By introducing the traceless field $\tilde{\mathbf{n}}(x)$ defined by [which we call the **color (direction) field** after the normalization]

$$\tilde{\mathbf{n}}(x) := g(x) \left[\rho - \frac{\mathbf{1}}{\text{tr}(\mathbf{1})} \right] g^\dagger(x) = g(x)\rho g^\dagger(x) - \frac{\mathbf{1}}{\text{tr}(\mathbf{1})}, \quad (26)$$

the ‘‘Abelian’’ field A^g is rewritten as

$$A_\mu^g(x) = \text{tr}\{\tilde{\mathbf{n}}(x)\mathcal{A}_\mu(x)\} + ig_{\text{YM}}^{-1} \text{tr}\{\rho g^\dagger(x)\partial_\mu g(x)\}. \quad (27)$$

We proceed to perform the **decomposition of the Yang-Mills field** $\mathcal{A}_\mu(x)$ into two pieces:

$$\mathcal{A}_\mu(x) = \mathcal{V}_\mu(x) + \mathcal{X}_\mu(x). \quad (28)$$

We simply require that $\mathcal{X}_\mu(x)$ satisfies the condition:[which we call the second **defining equation**]

$$(ii) \quad \mathcal{X}_\mu(x) \cdot \mathbf{n}(x) = 2\text{tr}\{\mathcal{X}_\mu(x)\mathbf{n}(x)\} = 0. \quad (29)$$

Then $\mathcal{X}_\mu(x)$ disappears from the Wilson loop operator, since $A_\mu^g(x)$ is written without $\mathcal{X}_\mu(x)$:

$$A_\mu^g(x) = \text{tr}\{\tilde{\mathbf{n}}(x)\mathcal{V}_\mu(x)\} + ig_{\text{YM}}^{-1} \text{tr}\{\rho g^\dagger(x)\partial_\mu g(x)\}. \quad (30)$$

Consequently, the Wilson loop operator $W_C[\mathcal{A}]$ can be reproduced by the **restricted field** variable $\mathcal{V}_\mu(x)$ alone. This is called the **restricted field dominance** for the Wilson loop operator. For arbitrary loop C and any representation R , the Wilson loop operators satisfies

$$(a) \quad W_C[\mathcal{A}] = W_C[\mathcal{V}]. \quad (31)$$

This does not necessarily imply the restricted field dominance for the Wilson loop average:

$$\langle W_C[\mathcal{A}] \rangle_{\text{YM}} = \langle W_C[\mathcal{V}] \rangle_{\text{YM}}, \quad (32)$$

which holds only when the cross term between \mathcal{V} and \mathcal{X} in the action can be neglected.

We look for the **gauge covariant decomposition** which means that the decomposition holds after the gauge transformation:

$$\mathcal{A}_\mu'(x) = \mathcal{V}_\mu'(x) + \mathcal{X}_\mu'(x). \quad (33)$$

For the condition (ii) [eq.(29)] to be gauge covariant, the transformation of the color field \mathbf{n} given by

$$g(x) \rightarrow U(x)g(x) \implies \mathbf{n}(x) \rightarrow \mathbf{n}'(x) = U(x)\mathbf{n}(x)U^\dagger(x). \quad (34)$$

requires that $\mathcal{X}_\mu(x)$ transforms as an adjoint (matter) field:

$$\mathcal{X}_\mu(x) \rightarrow \mathcal{X}_\mu'(x) = U(x)\mathcal{X}_\mu(x)U^\dagger(x), \quad (35)$$

This immediately means that $\mathcal{V}_\mu(x)$ must transform just like the original gauge field $\mathcal{A}_\mu(x)$:

$$\mathcal{V}_\mu(x) \rightarrow \mathcal{V}'_\mu(x) = U(x)\mathcal{V}_\mu(x)U^\dagger(x) + ig_{\text{YM}}^{-1}U(x)\partial_\mu U^\dagger(x), \quad (36)$$

since $\mathcal{A}_\mu(x) \rightarrow \mathcal{A}'_\mu(x) = U(x)\mathcal{A}_\mu(x)U^\dagger(x) + ig_{\text{YM}}^{-1}U(x)\partial_\mu U^\dagger(x)$.

These transformation properties impose restrictions on the requirement to be imposed on the restricted field $\mathcal{V}_\mu(x)$. Such a candidate is [covariant constantness of the color field] [which we call the first **defining equation**]:

$$(I) \quad \mathcal{D}_\mu[\mathcal{V}]\mathbf{n} = 0 \quad (\mathcal{D}_\mu[\mathcal{V}] := \partial_\mu - ig_{\text{YM}}[\mathcal{V}_\mu, \cdot]), \quad (37)$$

since the covariant derivative transforms in the adjoint way: $\mathcal{D}_\mu[\mathcal{V}(x)] \rightarrow U(x)(\mathcal{D}_\mu[\mathcal{V}](x))U^\dagger(x)$.

For $G = SU(2)$, it is shown that the two conditions (I) and (ii), i.e., (37) and (29) [the **defining equations** for the decomposition] are compatible and determine the decomposition uniquely:

$$\begin{aligned} \mathcal{A}_\mu(x) &= \mathcal{V}_\mu(x) + \mathcal{X}_\mu(x), \\ \mathcal{V}_\mu(x) &= c_\mu(x)\mathbf{n}(x) + ig_{\text{YM}}^{-1}[\mathbf{n}(x), \partial_\mu \mathbf{n}(x)], \quad c_\mu(x) := \mathcal{A}_\mu(x) \cdot \mathbf{n}(x), \\ \mathcal{X}_\mu(x) &= -ig_{\text{YM}}^{-1}[\mathbf{n}(x), \mathcal{D}_\mu[\mathcal{A}]\mathbf{n}(x)]. \end{aligned} \quad (38)$$

This is the same as the **Cho–Duan–Ge (CDG) decomposition** or **Cho–Duan–Ge–Faddeev–Niemi (CDGFN) decomposition** [Cho(1980), Duan-Ge (1979), Faddeev-Niemi (1998)] [8, 9, 10, 11].

The condition (I) means that the field strength $\mathcal{F}_{\mu\nu}^{[\mathcal{V}]}(x)$ of the field $\mathcal{V}_\mu(x)$ and $\mathbf{n}(x)$ commute:

$$[\mathcal{F}_{\mu\nu}^{[\mathcal{V}]}(x), \mathbf{n}(x)] = 0. \quad (39)$$

This follows from the identity:

$$[\mathcal{F}_{\mu\nu}^{[\mathcal{V}]}(x), \mathbf{n}(x)] = ig_{\text{YM}}^{-1}[\mathcal{D}_\mu^{[\mathcal{V}]}(x), \mathcal{D}_\nu^{[\mathcal{V}]}(x)]\mathbf{n}, \quad (40)$$

which is derived from

$$\mathcal{F}_{\mu\nu}^{[\mathcal{V}]}(x) = ig_{\text{YM}}^{-1}[\mathcal{D}_\mu^{[\mathcal{V}]}(x), \mathcal{D}_\nu^{[\mathcal{V}]}(x)], \quad \mathcal{D}_\mu^{[\mathcal{V}]}(x) := \partial_\mu - ig_{\text{YM}}[\mathcal{V}_\mu, \cdot]. \quad (41)$$

For $SU(2)$, (39) means that $\mathcal{F}_{\mu\nu}^{[\mathcal{V}]}(x)$ is proportional to $\mathbf{n}(x)$:

$$\mathcal{F}_{\mu\nu}^{[\mathcal{V}]}(x) = f_{\mu\nu}(x)\mathbf{n}(x) \implies f_{\mu\nu}(x) = \mathbf{n}(x) \cdot \mathcal{F}_{\mu\nu}^{[\mathcal{V}]}(x) = 2\text{tr}[\mathbf{n}(x)\mathcal{F}_{\mu\nu}^{[\mathcal{V}]}(x)], \quad (42)$$

since $\mathcal{F}_{\mu\nu}^{[\mathcal{V}]}(x)$ is traceless and cannot have a part proportional to the unit matrix.

FIELD DECOMPOSITION FOR $SU(N)$: NEW OPTIONS

For $G = SU(N)$ ($N \geq 3$), (I) and (ii) are not sufficient to uniquely determine the decomposition. The condition (ii) [eq.(29)] must be modified: [Kondo, Shinohara and Murakami (2008)][25]

(II) $\mathcal{X}^\mu(x)$ does not have the \tilde{H} -commutative part, i.e., $\mathcal{X}^\mu(x)\tilde{H} = 0$:

$$\begin{aligned} (II) \quad 0 &= \mathcal{X}^\mu(x)\tilde{H} := \mathcal{X}^\mu(x) - \frac{2(N-1)}{N}[\mathbf{n}(x), [\mathbf{n}(x), \mathcal{X}^\mu(x)]] \\ &\iff \mathcal{X}^\mu(x) = \frac{2(N-1)}{N}[\mathbf{n}(x), [\mathbf{n}(x), \mathcal{X}^\mu(x)]]. \end{aligned} \quad (43)$$

This condition is also gauge covariant. Note that the condition (ii)[eq.(29)] follows from (II)[eq.(43)]. For $G = SU(2)$, i.e., $N = 2$, the condition (II)[eq.(43)] reduces to (ii)[eq.(29)]. By solving (I)[eq.(37)] and (II)[eq.(43)], $\mathcal{X}_\mu(x)$ is determined as

$$\mathcal{X}_\mu(x) = -ig_{\text{YM}}^{-1}\frac{2(N-1)}{N}[\mathbf{n}(x), \mathcal{D}_\mu[\mathcal{A}]\mathbf{n}(x)] \in \text{Lie}(G/\tilde{H}), \quad (44)$$

$$\mathcal{V}_\mu(x) = \mathcal{C}_\mu(x) + \mathcal{B}_\mu(x) \in \mathcal{L}ie(G),$$

$$\mathcal{C}_\mu(x) = \mathcal{A}_\mu(x) - \frac{2(N-1)}{N}[\mathbf{n}(x), [\mathbf{n}(x), \mathcal{A}_\mu(x)]] \in \mathcal{L}ie(\tilde{H}),$$

$$\mathcal{B}_\mu(x) = ig_{\text{YM}}^{-1}\frac{2(N-1)}{N}[\mathbf{n}(x), \partial_\mu \mathbf{n}(x)] \in \mathcal{L}ie(G/\tilde{H}). \quad (45)$$

NON-ABELIAN STOKES THEOREM (2)

Finally, we can show that the field strength $F_{\mu\nu}^g := \partial_\mu A_\nu - \partial_\nu A_\mu$ in NAST (20) is cast into the form:

$$F_{\mu\nu}^g(x) = \sqrt{\frac{2(N-1)}{N}} \text{tr}\{\mathbf{n}(x) \mathcal{F}_{\mu\nu}[\mathcal{V}](x)\} + ig_{\text{YM}}^{-1} \text{tr}\{\rho g^\dagger(x) [\partial_\mu, \partial_\nu] g(x)\}. \quad (46)$$

$$\text{tr}\{\mathbf{n}(x) \mathcal{F}_{\mu\nu}[\mathcal{V}](x)\} = \partial_\mu \text{tr}\{\mathbf{n}(x) \mathcal{V}_\nu(x)\} - \partial_\nu \text{tr}\{\mathbf{n}(x) \mathcal{V}_\mu(x)\} + \frac{2(N-1)}{N} ig_{\text{YM}}^{-1} \text{tr}\{\mathbf{n}(x) [\partial_\mu \mathbf{n}(x), \partial_\nu \mathbf{n}(x)]\}, \quad (47)$$

where the normalized and traceless **color direction field** $\mathbf{n}(x)$ is defined by

$$\mathbf{n}(x) = \sqrt{\frac{N}{2(N-1)}} g(x) \left[\rho - \frac{\mathbf{1}}{\text{tr}(\mathbf{1})} \right] g^\dagger(x), \quad g(x) \in G. \quad (48)$$

Thus the Wilson loop operator can be rewritten in terms of new variables:

$$W_C[\mathcal{A}] = \int [d\mu(g)]_{\Sigma_C} \exp \left[-ig_{\text{YM}} \frac{1}{2} \sqrt{\frac{2(N-1)}{N}} \int_{\Sigma_C: \partial \Sigma_C = C} 2\text{tr}\{\mathbf{n} \mathcal{F}[\mathcal{V}]\} \right]. \quad (49)$$

Incidentally, the last part $ig_{\text{YM}}^{-1} \text{tr}\{\rho g(x)^\dagger [\partial_\mu, \partial_\nu] g(x)\}$ in $F_{\mu\nu}^g(x)$ corresponds to the **Dirac string**. This term is not gauge invariant and does not contribute to the Wilson loop operator in the end, since it disappears after the group integration $d\mu(g)$ is performed.

In this way we obtain another expression of the NAST for the Wilson loop operator: For $SU(N)$ in the **fundamental representation**:

$$W_C[\mathcal{A}] = \int [d\mu(g)] \exp \left\{ -ig_{\text{YM}} \frac{1}{2} \sqrt{\frac{2(N-1)}{N}} [(N_{\Sigma_C}, j) + (\Xi_{\Sigma_C}, k)] \right\}, \quad (50)$$

where we have defined the $(D-3)$ -form k and one-form j by

$$k := \delta^* f, \quad j := \delta f, \quad f := 2\text{tr}\{\mathbf{n} \mathcal{F}[\mathcal{V}]\}, \quad (51)$$

and we have defined the $(D-3)$ -form Ξ_{Σ_C} and one-form N_{Σ_C} by (Ξ_{Σ_C} is the D -dim. solid angle)

$$\Xi_{\Sigma_C} := *d\Delta^{-1} \Theta_{\Sigma_C} = \delta \Delta^{-1} * \Theta_{\Sigma_C}, \quad N_{\Sigma_C} := \delta \Delta^{-1} \Theta_{\Sigma_C}, \quad (52)$$

with the inner product for the two forms defined by

$$(\Xi_{\Sigma_C}, k) = \frac{1}{(D-3)!} \int d^D x k^{\mu_1 \dots \mu_{D-3}}(x) \Xi_{\Sigma_C}^{\mu_1 \dots \mu_{D-3}}(x), \quad (N_{\Sigma_C}, j) = \int d^D x j^\mu(x) N_{\Sigma_C}^\mu(x). \quad (53)$$

Thus the Wilson loop operator can be expressed by the electric current j and the monopole current k .

The magnetic monopole described by the current k is a topological object of **co-dimension 3**:

- $D = 3$: 0-dimensional point defect \rightarrow point-like magnetic monopole (cf. Wu-Yang type)
- $D = 4$: 1-dimensional line defect \rightarrow magnetic monopole loop (closed loop)

$SU(2)$ case

For $SU(2)$, the gauge-invariant magnetic-monopole current $(D-3)$ -form k is obtained

$$k = \delta^* f, \quad f_{\mu\nu} = 2\text{tr}\{\mathbf{n} \mathcal{F}_{\mu\nu}[\mathcal{V}]\} = \partial_\mu 2\text{tr}\{\mathbf{n} \mathcal{A}_\nu\} - \partial_\nu 2\text{tr}\{\mathbf{n} \mathcal{A}_\mu\} + ig_{\text{YM}}^{-1} 2\text{tr}\{\mathbf{n} [\partial_\mu \mathbf{n}, \partial_\nu \mathbf{n}]\}. \quad (54)$$

For the fundamental representation of $SU(2)$, the highest-weight state $|\Lambda\rangle$ yields the color field:

$$\begin{aligned} |\Lambda\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \rho := |\Lambda\rangle\langle\Lambda| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \rho - \frac{1}{2} \mathbf{1} = \frac{\sigma_3}{2}, \\ &\Rightarrow \mathbf{n}(x) = g(x) \frac{\sigma_3}{2} g(x)^\dagger \in SU(2)/U(1) \simeq S^2 \simeq P^1(\mathbb{C}). \end{aligned} \quad (55)$$

The magnetic charge q_m obeys the **quantization condition** à la Dirac:

$$q_m := \int d^3 x k^0 = 4\pi g_{\text{YM}}^{-1} \ell, \quad \ell \in \mathbb{Z}. \quad (56)$$

This is suggested from a nontrivial Homotopy group of the map $\mathbf{n} : S^2 \rightarrow SU(2)/U(1)$:

$$\pi_2(SU(2)/U(1)) = \pi_1(U(1)) = \mathbb{Z}. \quad (57)$$

This should be compared with the Abelian magnetic monopole due to 't Hooft-Polyakov associated with the spontaneous symmetry breaking $G = SU(2) \rightarrow H = U(1)$:

$$\mathbf{n}^A \leftrightarrow \hat{\phi}^A(x)/|\hat{\phi}(x)|. \quad (58)$$

$SU(3)$ case

For $SU(3)$, the gauge-invariant magnetic-monopole current $(D-3)$ -form k is given by [Kondo (2008)][22]

$$k = \delta^* f, \quad f_{\mu\nu} := \partial_\mu 2\text{tr}\{\mathbf{n}\mathcal{A}_\nu\} - \partial_\nu 2\text{tr}\{\mathbf{n}\mathcal{A}_\mu\} + \frac{4}{3}ig_{\text{YM}}^{-1}2\text{tr}\{\partial_\mu \mathbf{n}, \partial_\nu \mathbf{n}\}. \quad (59)$$

For the fundamental representation of $SU(3)$, the highest-weight state $|\Lambda\rangle$ yields the color field:

$$|\Lambda\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \rho := |\Lambda\rangle\langle\Lambda| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \rho - \frac{1}{3}\mathbf{1} = \frac{-1}{3} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (60)$$

$$\Rightarrow \mathbf{n}(x) = g(x) \frac{-1}{2\sqrt{3}} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g(x)^\dagger \in SU(3)/U(2) \simeq P^2(\mathbb{C}). \quad (61)$$

The matrix $\text{diag.}(-2, 1, 1)$ is degenerate. Using the Weyl symmetry (discrete global symmetry as a subgroup of color symmetry), it is changed into λ_8 . This color field describes a **non-Abelian magnetic monopole**, which corresponds to the spontaneous symmetry breaking $SU(3) \rightarrow U(2)$ in the gauge-Higgs model.

The magnetic charge obeys the quantization condition:

$$q'_m := \int d^3x k^0 = 2\pi\sqrt{3}g_{\text{YM}}^{-1}n', \quad n' \in \mathbb{Z}. \quad (62)$$

This is suggested from a nontrivial Homotopy group of the map $\mathbf{n} : S^2 \rightarrow SU(3)/U(2)$

$$\pi_2(SU(3)/[SU(2) \times U(1)]) = \pi_1(SU(2) \times U(1)) = \pi_1(U(1)) = \mathbb{Z}. \quad (63)$$

For a **reference state** $|\Lambda\rangle$ of a given representation of a Lie group G , the **maximal stability subgroup** \tilde{H} is defined to be a subgroup leaving $|\Lambda\rangle$ invariant (up to a phase $\phi(h)$):

$$h \in \tilde{H} \iff h|\Lambda\rangle = |\Lambda\rangle e^{i\phi(h)}. \quad (64)$$

Then a group element g of G is decomposed as

$$g = \xi h \in G, \quad \xi \in G/\tilde{H}, \quad h \in \tilde{H}. \quad (65)$$

Therefore, we have

$$|g, \Lambda\rangle := g|\Lambda\rangle = \xi h|\Lambda\rangle = \xi|\Lambda\rangle e^{i\phi(h)} = |\xi, \Lambda\rangle e^{i\phi(h)}. \quad (66)$$

Every representation R of $SU(3)$ which is specified by the Dynkin index $[m, n]$ belongs to (I) or (II):

(I) [Maximal case] $m \neq 0$ and $n \neq 0 \implies \tilde{H} = H = U(1) \times U(1)$. maximal torus

e.g., adjoint rep. $[1, 1]$, $\{H_1, H_2\} \in u(1) + u(1)$,

(II) [Minimal case] $m = 0$ or $n = 0 \implies \tilde{H} = U(2)$.

This case occurs when the weight vector Λ is orthogonal to some of the root vectors. (See Fig. 5)

e.g., fundamental rep. $[1, 0]$, $\{H_1, H_2, E_\beta, E_{-\beta}\} \in u(2)$, where $\Lambda \perp \beta, -\beta$.

TABLE 1.

	original YM	\implies reformulated YM
field variables	$\mathcal{A}_\mu^A \in \mathcal{L}(G)$	$\implies \mathbf{n}^\beta, \mathcal{C}_V^k, \mathcal{X}_V^b$
action	$S_{\text{YM}}[\mathcal{A}]$	$\implies \tilde{S}_{\text{YM}}[\mathbf{n}, \mathcal{C}, \mathcal{X}]$
integration measure	$\mathcal{D}\mathcal{A}_\mu^A$	$\implies \mathcal{D}\mathbf{n}^\beta \mathcal{D}\mathcal{C}_V^k \mathcal{D}\mathcal{X}_V^b \tilde{J}\delta(\tilde{\chi})\Delta_{\text{FP}}^{\text{red}}[\mathbf{n}, c, \mathcal{X}]$

In fact, the enlarged gauge transformation of the functional $F_{\text{red}}[\mathcal{A}, \mathbf{n}]$ is

$$\delta F_{\text{red}}[\mathcal{A}, \mathbf{n}] = \delta \int d^D x \frac{1}{2} (D_\mu[\mathcal{A}]\mathbf{n})^2 = g \int d^D x (\boldsymbol{\omega}_\perp - \boldsymbol{\omega}_\perp) \cdot i[\mathbf{n}, D^\mu[\mathcal{A}]D_\mu[\mathcal{A}]\mathbf{n}], \quad (70)$$

where $\boldsymbol{\omega}_\perp$ denotes the component of $\boldsymbol{\omega}$ in the direction $\mathcal{L}(G/\tilde{H})$.

For $\boldsymbol{\omega}_\perp = \boldsymbol{\theta}_\perp$ (diagonal part of $G \times G/\tilde{H}$) $\delta F_{\text{red}}[\mathcal{A}, \mathbf{n}] = 0$ imposes no condition, while for $\boldsymbol{\omega}_\perp \neq \boldsymbol{\theta}_\perp$ (off-diagonal part of $G \times G/\tilde{H}$) it implies the constraint:

$$\chi[\mathcal{A}, \mathbf{n}] := [\mathbf{n}, D^\mu[\mathcal{A}]D_\mu[\mathcal{A}]\mathbf{n}] \equiv 0. \quad (71)$$

Note that the number of constraint is $\#\chi = \dim(G \times G/\tilde{H}) - \dim(G) = \dim(G/\tilde{H})$ as desired.

Finally, we have an equipollent Yang-Mills theory with **the residual local gauge symmetry** $G' := SU(N)_{\omega'}^{\text{local}}$ with the gauge transformation parameter:

$$\omega'(x) = (\boldsymbol{\omega}_\parallel(x), \boldsymbol{\omega}_\perp(x)) = (\boldsymbol{\omega}_\parallel(x), \boldsymbol{\theta}_\perp(x)), \quad \boldsymbol{\omega}_\perp(x) = \boldsymbol{\theta}_\perp(x). \quad (72)$$

At the same time, the color field

$$\mathbf{n}(x) \in \mathcal{L}ie(G/\tilde{H}) \quad (73)$$

must be obtained by solving the reduction condition $\chi = 0$ for a given \mathcal{A} , e.g.,

$$\chi[\mathcal{A}, \mathbf{n}] := [\mathbf{n}, D^\mu[\mathcal{A}]D_\mu[\mathcal{A}]\mathbf{n}] \in \mathcal{L}ie(G/\tilde{H}). \quad (74)$$

Here $\tilde{\chi} = 0$ is the reduction condition written in terms of the new variables:

$$\tilde{\chi} := \tilde{\chi}[\mathbf{n}, \mathcal{C}, \mathcal{X}] := D^\mu[\mathcal{V}]\mathcal{X}_\mu, \quad (75)$$

and $\Delta_{\text{FP}}^{\text{red}}$ is the Faddeev-Popov determinant associated with the reduction condition:

$$\Delta_{\text{FP}}^{\text{red}} := \det \left(\frac{\delta \chi}{\delta \boldsymbol{\theta}} \right)_{\chi=0} = \det \left(\frac{\delta \chi}{\delta \mathbf{n}^\theta} \right)_{\chi=0}, \quad (76)$$

which is obtained by the BRST method as $\Delta_{\text{FP}}^{\text{red}}[\mathbf{n}, c, \mathcal{X}] = \det\{-D_\mu[\mathcal{V} + \mathcal{X}]D_\mu[\mathcal{V} - \mathcal{X}]\}$. The Jacobian \tilde{J} is very simple, irrespective of the choice of the reduction condition: [14, 32]

$$\tilde{J} = 1. \quad (77)$$

Thus the Wilson loop average in the original theory defined by

$$W(C) := \langle W_C[\mathcal{A}] \rangle_{\text{YM}} = Z_{\text{YM}}^{-1} \int \mathcal{D}\mathcal{A} e^{-S_{\text{YM}}[\mathcal{A}]} W_C[\mathcal{A}], \quad (78)$$

is defined in the reformulated Yang-Mills theory:

$$\begin{aligned} \langle W_C[\mathcal{A}] \rangle_{\text{YM}'} &= Z_{\text{YM}'}^{-1} \int [d\mu(g)] \int \mathcal{D}\mathbf{n}^\beta \mathcal{D}\mathcal{C}_V^k \mathcal{D}\mathcal{X}_V^b \tilde{J}\delta(\tilde{\chi})\Delta_{\text{FP}}^{\text{red}} e^{-\tilde{S}_{\text{YM}}[\mathbf{n}, \mathcal{C}, \mathcal{X}]} \exp \left\{ i g_{\text{YM}} \sqrt{\frac{2(N-1)}{N}} [(j, N_{\Sigma_C}) + (k, \Xi_{\Sigma_C})] \right\}, \\ Z_{\text{YM}'} &= \int \mathcal{D}\mathbf{n}^\beta \mathcal{D}\mathcal{C}_V^k \mathcal{D}\mathcal{X}_V^b \tilde{J}\delta(\tilde{\chi})\Delta_{\text{FP}}^{\text{red}} e^{-S_{\text{YM}'}[\mathbf{n}, \mathcal{C}, \mathcal{X}]}. \end{aligned} \quad (79)$$

Remark:

1. For $SU(2)$, when we fix the color field $\mathbf{n}(x) = (0, 0, 1)$ or $\mathbf{n}(x) = \sigma_3/2$, the reduction condition $D^\mu[\mathcal{V}]\mathcal{X}_\mu = 0$ reduces to the conventional Maximally Abelian gauge [5].
2. For $SU(3)$, this is not the case: This reduction does not reduce to the conventional Maximally Abelian gauge for $SU(3)$, even if the color field is fixed to be uniform. Therefore, the results to be obtained are nontrivial.

CONCLUSION AND DISCUSSION

We have combined a non-Abelian Stokes theorem for the Wilson loop operator [22] and the new reformulations of the Yang-Mills theory [25] to study quark confinement from a viewpoint of the dual superconductor. The obtained results are summarized as follows.

- 1) In order to define (chromo)magnetic monopoles in the $SU(N)$ Yang-Mills theory (without adjoint scalar fields), we do not need to use the prescription called the **Abelian projection** [’t Hooft, 1981][4] which realizes magnetic monopoles as **gauge-fixing defects**. In fact, we can define gauge-invariant magnetic monopoles which are inherent in the non-Abelian Wilson loop operator and we can extract them by using a **non-Abelian Stokes theorem** for the non-Abelian Wilson loop operator [22].
- 2) For the $G = SU(2)$ gauge group, the resulting magnetic monopole coincide with one obtained from the CDG decomposition for the Yang-Mills field which was proposed by [Cho (1980)] and [Duan & Ge (1979)] independently. For the $G = SU(2)$ gauge group, such an Abelian magnetic monopole is described by the color field $\mathbf{n}(x)$ with the target space: $\mathbf{n}(x) \in SU(2)/U(1) = P^1(\mathbb{C})$ for quarks in any representation. However, $G = SU(2)$ is an exceptional case.
- 3) For $SU(N)$ ($N \geq 3$), the resulting magnetic monopole depends on the representation of quarks defining the Wilson loop operator, which is related to the specific target space of the color field $\mathbf{n}(x)$. For the $G = SU(3)$ gauge group, every representation of $SU(3)$ is specified by the Dynkin index $[m, n]$ and the magnetic monopoles are exhausted by two cases:
 - For quarks in the representation $m = 0$ or $n = 0$, $\tilde{H} = U(2)$, e.g., the fundamental representation of $G = SU(3)$, a non-Abelian magnetic monopole described by $\mathbf{n}(x) \in SU(3)/U(2) = P^2(\mathbb{C})$
 - For quarks in the representation $m \neq 0$ and $n \neq 0$, $\tilde{H} = H = U(1) \times U(1)$, e.g., the adjoint representation of $G = SU(3)$, two Abelian magnetic monopoles described by $\mathbf{n}(x) \in SU(3)/[U(1) \times U(1)] = F_2$
 Here \tilde{H} is a subgroup of G called the maximal stability group which is uniquely determined once the representation is specified. \tilde{H} does not necessarily agree with the maximal torus group $H = U(1)^{N-1}$.

- 4) We have constructed the new reformulations of the $SU(N)$ Yang-Mills theory using new field variables so that they give the optimal description of the gauge-invariant magnetic monopole defined through the $SU(N)$ Wilson loop operator. This is an extension of the work due to [Cho (1980)][23] and [Faddeev & Niemi (1999)][24] including their results as a special option where $N - 1$ color fields $\mathbf{n}^{(j)}(x)$ ($j = 1, \dots, N - 1$) corresponding all the Cartan subgroup are introduced. The reformulation allows a number of options discriminated by the maximal stability group \tilde{H} of the gauge group G . Our reformulations introduce only a single color field $\mathbf{n}(x)$ for any N , which is enough for reformulating the quantum Yang-Mills theory to describe confinement of the fundamental quark. For $G = SU(3)$, two options are possible:
 - The maximal option with $\tilde{H} = H = U(1) \times U(1)$, the reformulation gives a manifestly gauge-independent extension of the conventional Abelian projection in the maximal Abelian gauge. This is just the case of Cho and Faddeev & Niemi.
 - The minimal option with $\tilde{H} = U(2)$ gives an optimized description of quark confinement through the Wilson loop operator in the fundamental representation. [Kondo, Shinohara and Murakami, 2008] The minimal option in our reformulation is new for $SU(N)$, $N \geq 3$:

- 5) Moreover, we have constructed the lattice versions[15, 16, 17] [26, 27] of the reformulations of the $SU(N)$ Yang-Mills theory and performed numerical simulations on a lattice for $SU(2)$ and $SU(3)$. [Talk by Akihiro Shibata, see the contribution to this conference.]

5a) For $SU(2)$ and $SU(3)$, we have confirmed the infrared restricted field dominance and the magnetic monopole dominance for quark confinement:

- For $SU(2)$, we have confirmed the infrared dominance of the restricted variables \mathcal{V} (a gauge-independent version of the “Abelian” dominance) and the Abelian magnetic monopole dominance for confinement of quark (in the string tension) in any representation. [15, 16]
cf. [infrared Abelian dominance and Abelian magnetic monopole dominance in the MA gauge (Abelian projection)]
- For $SU(3)$, we have confirmed the restricted field dominance \mathcal{V} and the non-Abelian magnetic monopole dominance for confinement of quark (in the string tension) in the fundamental representation. [29]

For $SU(2)$ and $SU(3)$, we have presented the suppression of the remaining field \mathcal{X} (exponential fall-off of the correlation function) in the low-energy or the long distance region. [17]

5b) For $SU(2)$ and $SU(3)$, we have given the numreical evidences for the dual Meissner effect caused by gauge-invariant magnetic monopoles in the Yang-Mills theory: simultaneous formation of the chromoelectric flux tube connecting a pair of quark and antiquark, and the magnetic current induced around the flux tube [30].

For $SU(2)$ and $SU(3)$, we have confirmed also the infrared restricted field dominance in the dual Meissner effect suggesting the magnetic monopole condensations.

Moreover, we have determined the type of the dual superconductivity by measuring the penetration depth and the coherent length (assuming the relativistic Ginzburg-Landau model for fitting the data). [31, 30]

- For $SU(2)$, the type of the dual superconductivity is the border between type I and II or rather weakly type I [31]. This is consistent with the preceding works.
- For $SU(3)$, the type is strictly type I [30]. This is a new result which is consistent with the result of other groups.

These results support the non-Abelian dual superconductivity picture for quark confinement in $SU(3)$ Yang-Mills theory.

For applications of the reformulation to other topics, see the recent review [32].

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